Accelerated Direct Solution of the MoM-VIE for Dielectric Scatterers

Alex Heldring, José Maria Tamayo, Juan Manuel Rius, *Member, IEEE* and Eduard Ubeda

***Abstract*—The Multiscale Compressed Block Decomposition algorithm (MS-CBD), a direct (non-iterative) linear solver, is applied to accelerate the solution of the MoM-VIE formulation for dielectric scatterers. Numerical solutions are presented for problems with several hundreds of thousands of unknowns. Asymptotically (with respect to the electrical size of the problem), the solution time scales with the number of unknowns squared. The numerical examples confirm this theoretical value.**

***Index Terms*— Numerical Electromagnetics, Dielectrics, Method of Moments, Volume Integral Equation, Fast Methods.**

I. INTRODUCTION

THE Volume Integral Equation (VIE) formulation of the Method of Moments for scattering and radiation problems involving dielectric objects has some very attractive features in comparison with Surface Integral Equation (SIE) formulations. In particular, the spatial distribution of the permittivity can be arbitrarily complex without any penalty for the computations. Furthermore, the formulation is robust, yielding a well conditioned linear system, irrespective of the problem frequency. There is, however, an important penalty to pay: the formulation requires volumetric discretization with typically approximately 10 samples per wavelength inside the dielectric medium. This means that the necessary number of basis functions N grows very fast for electrically large objects. And since the MoM yields a full impedance matrix, the storage requirements grow with N^2 and straightforward solution by LU decomposition of the linear system grows with N^3.

It is therefore no surprise that several authors have proposed to combine the VIE with the same kind of accelerating techniques that were originally developed for SIE formulations but are readily adaptable to the VIE. Examples are the MLFMA [1], the AIM [2] and the P-FFT [3] methods. These accelerating techniques have in common an operator that approximates the MoM impedance matrix with much reduced storage and complexity, and which can be used within an iterative solver.

At AntennaLab we have recently developed an algorithm which, unlike the above-mentioned methods, yields a direct (non-iterative) solution of the MoM linear system, with much reduced storage and complexity. This method, the Compressed Block Decomposition (CBD) [4], and its Multiscale version, the MS-CBD [5], is based on block wise subdivision of the problem geometry, and compression of the resulting impedance matrix sub blocks using the ACA algorithm [6] and thresholded SVD. The compressed matrix is then decomposed, preserving the initial compression, into factors that allow fast solution for one or many independent vectors by back-substitution.

Both algorithms have been extensively tested on problems involving perfectly conducting surfaces, using surface discretization and the EFIE formulation. They have proven to highly reduce both solution time and storage requirements with respect to ordinary LU decomposition, with little and controllable loss of accuracy. Problems involving hundreds of thousands of unknowns, even up to one million, and thousands of independent vectors, are solved on an ordinary PC in hours.

Although iterative methods are still considerably faster than the MS-CBD, there are a number of reasons why under some circumstances a direct method such as MS-CBD may be preferable: direct methods are more robust and do not need a preconditioner. Also, direct methods are more efficient in dealing with multiple excitations (eg. Monostatic RCS computations).

This paper presents the MS-CBD algorithm applied to the VIE for dielectric objects.

II. VIE

We have chosen to implement the VIE formulation introduced by Schaubert, Wilton and Glisson in 1984 [7], because it is commonly used and therefore thoroughly tested, it is versatile due to the tetrahedral discretization, and it is well suited to combine with the well known RWG basis functions which we intend to do in the near future.

The VIE is given by

$$\boldsymbol{D}/\varepsilon = \boldsymbol{E}_{inc} - j\omega\boldsymbol{A} - \nabla\Phi, \tag{1}$$

in which ε is the permittivity of the medium, $\boldsymbol{D}$ is the

This work was supported by the Spanish Interministerial Commission on Science and Technology (CICYT) under projects TEC2006-13248-C04-02/TCM, TEC2006-13248-C04-01/TCM and TEC2007-66698-C04-01/TCM and CONSOLIDER CSD2008-00068 and by the "Ministerio de Educación y Ciencia" through the FPU fellowship program.

The authors are with AntennaLab, Dept. of Signal Theory and Communications, Polytechnic University of Catalunya, C. Jordi Girona, 31, 08034, Barcelona, Spain (email heldring@tsc.upc.edu).

unknown electric flux density and $\boldsymbol{E}_{inc}$ is the impressed incident field. $\boldsymbol{A}$ and $\boldsymbol{\Phi}$ are the usual vector and scalar potentials due to a polarization current density $\boldsymbol{J}$ which is related to $\boldsymbol{D}$ through

$$\boldsymbol{J} = j\omega(\varepsilon-\varepsilon_0)\boldsymbol{D}/\varepsilon. \quad (2)$$

Since the VIE is based on the equivalence principle, whereby the dielectric medium is replaced with equivalent polarization currents, the Greens function relating $\boldsymbol{A}$ and $\boldsymbol{\Phi}$ with $\boldsymbol{J}$ is that of free space.

In the model, $\boldsymbol{D}$ is represented by basis functions associated with every triangular face between tetrahedrons and on the outer surface of the model. $\boldsymbol{D}$ varies linearly inside every pair of (or single) tetrahedron associated with the face. Every tetrahedron is assigned a given constant permittivity. Once $\boldsymbol{D}$ is solved for, $\boldsymbol{J}$ is found from (2) and used to find the scattered fields or RCS.

III. MS-CBD

Applying the MoM to (1) leads to a linear system

$$\mathbf{E}_{\mathrm{inc}} = Z\,\mathbf{J} \quad (3)$$

To be solved for **J**. The MS-CBD is a nested implementation of the well known Partitioned Matrix Inverse formulas:

$$Z^{-1} = \begin{bmatrix} \tilde{Z}_{11} & \tilde{Z}_{12} \\ \tilde{Z}_{21} & \tilde{Z}_{11} \end{bmatrix}, \quad (4)$$

with

$$\begin{aligned} \tilde{Z}_{11} &= Z_{11}^{-1} + Z_{11}^{-1} Z_{12} \tilde{Z}_{22} Z_{21} Z_{11}^{-1} \\ \tilde{Z}_{12} &= -Z_{11}^{-1} Z_{12} \tilde{Z}_{22} \\ \tilde{Z}_{21} &= -\tilde{Z}_{22} Z_{21} Z_{11}^{-1} \\ \tilde{Z}_{22} &= (Z_{22} - Z_{21} Z_{11}^{-1} Z_{12})^{-1} \end{aligned}, \quad (5)$$

in which the partitioning is done according to geometrical grouping of the basis functions. This means that non-diagonal sub blocks represent interactions between geometrically separate groups of basis functions, which are generally low rank and can be compressed through truncated SVD. Because the SVD is an expensive algorithm the sub blocks are pre-compressed using the fast approximate ACA algorithm, followed by a post-compression with truncated SVD to achieve an optimum compression rate. The details of the MS-CBD will appear in [5].

IV. COMPLEXITY

In [5] we show that the storage requirements of the MS-CBD method, applied to MoM-SIE scale with $N^{1.5}$ as opposed to N^2 for the full impedance matrix. The complexity of the MS-CBD scales with N^2. It is easy to show that this result remains true for the VIE applied to an asymptotically large dielectric volume. These values apply to the case of fixed discretization size with respect to the frequency and are therefore valid for electrically large volumes. In practise it will often be more efficient to use an SIE formulation in such cases. The VIE is more suited to small, inhomogeneous geometries, for example printed circuits. For such problems, if a highly accurate solution is required, the discretization mesh must be refined, which is very costly for the VIE. Fortunately, since the MS-CBD compression rate does not depend on the mesh size, only on the electrical size of the geometry, the MS-CBD has linear storage and complexity for fixed frequency. Furthermore, the compression rate does not depend on the dielectric constant, since the Greens function in (1) is that of free space. Therefore, although a high permittivity medium of course requires a finer mesh, the compression rate is unaffected. In the following, we illustrate the three concepts described above which affect the efficiency of the VIE with MS-CBD numerically.

A. Complexity versus mesh size

In the literature, a discretization of $\lambda/10$ is recommended, which is also the typical value for surface discretization. However, very few examples of computations on bulk dielectric material are found in the published literature, so we chose to investigate the validity of this 'rule of thumb'.

To this aim we computed the RCS of a homogeneous dielectric sphere, with $\varepsilon_r = 4$ and radius *1m*, at *150 MHz* (such that λ inside the sphere equals *1m*), for various mesh-sizes.

The result is shown in Fig. 1. Clearly, for a good accuracy, a mesh size of approximately $\lambda/10$ is indeed necessary.

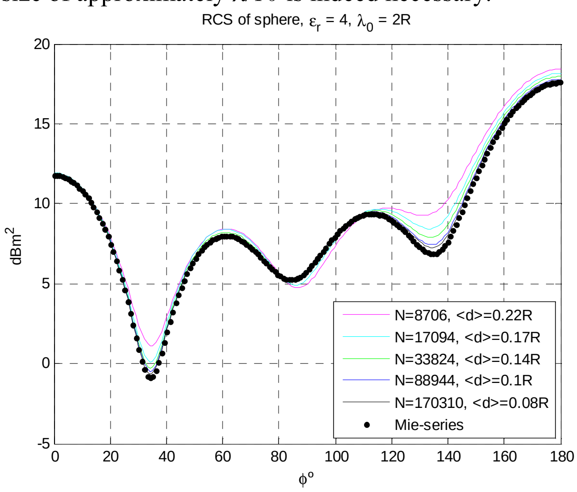

Fig. 1. RCS of dielectric sphere with radius R for different mesh sizes, analysed with MS-CBD (4-8 levels, $\tau = 10^{-3}$). N is the number of unknowns, <d> is the corresponding average tetrahedron edge length. The wavelength inside the sphere equals 1R.

Fortunately using MS-CBD the computational requirements grow slowly with *N*. In theory they are proportional to *N* but this is true for asymptotically large problems. For the above experiment they are somewhat larger, as shown in Figs 2 and 3.

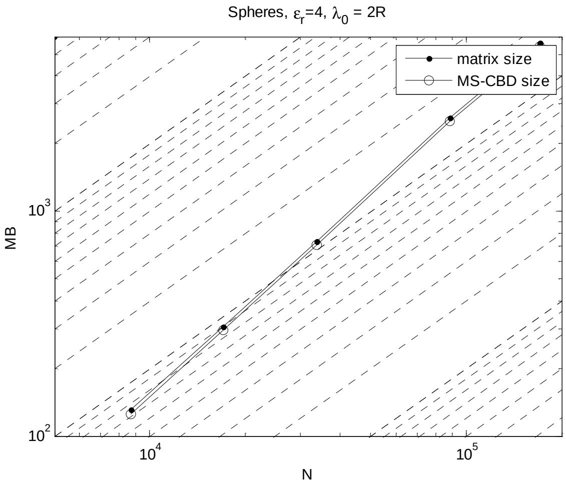

Fig. 2. Compressed matrix size and MS-CBD factorization size versus number of unknowns for fixed frequency. The dashed lines have linear slope.

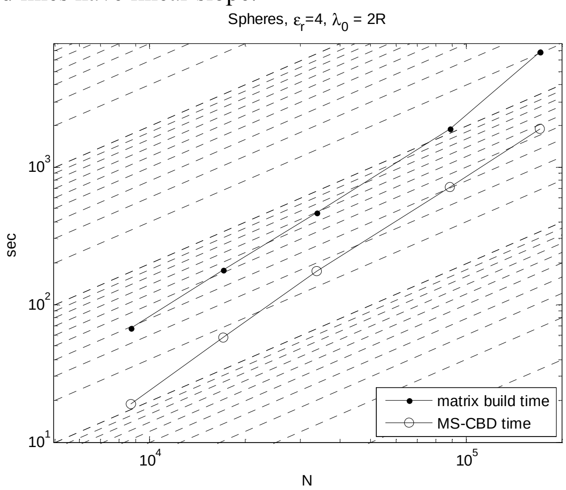

Fig. 3. Compressed matrix build time and MS-CBD factorization time versus number of unknowns for fixed frequency. The dashed lines have linear slope.

B. Complexity versus frequency

Usually one fixes the mesh size relative to the wave length to achieve a desired accuracy in the result. In electrodynamics, this leads to the common definition of the computational complexity, as the relation between the computational effort and *N*, with *N*, in the VIE formulation, being proportional to the cube of the frequency. We repeated the experiment from the previous section but now scaling the frequency to obtain a fixed average mesh size of 0.14λ for every sphere (λ being the wave length inside the sphere).

Figs. 4 and 5 show the results. One observes that both storage and complexity are actually somewhat better than the theoretical predictions from [5].

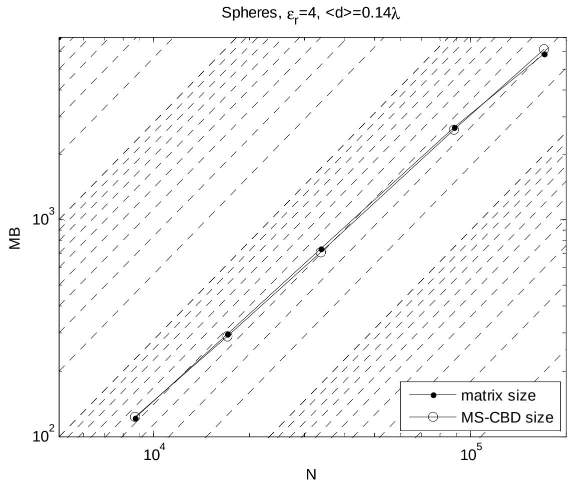

Fig. 4. Compressed matrix size and MS-CBD factorization size versus number of unknowns for fixed mesh size. The dashed lines are proportional to $N^{1.5}$.

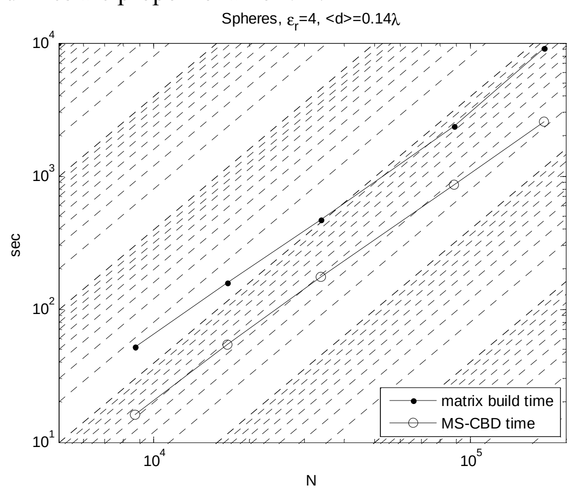

Fig. 5. Compressed matrix build time and MS-CBD factorization time versus number of unknowns for fixed mesh size. The dashed lines are proportional to N^2.

C. Complexity versus permittivity

In order to verify the claim that the efficiency of MS-CBD is independent of the permittivity, we repeated the experiment above for the N=33,824 sphere, λ=2R, for two different values of the relative permittivity of the sphere. The performances are shown in Table I. They corroborate the claim. Of course, for equal accuracy, the higher permittivity sphere would need more unknowns but this is unrelated to the MS-CBD algorithm.

Table I. Performance of MS-CBD versus ε_r.

N	33,824	
MS-CBD levels	6	
ε_r	4	9
Matrix build time	467 sec	496 sec
MS-CBD time	178 sec	169 sec
Matrix size	732 MB	733 MB
MS-CBD size	706 MB	651 MB

V. Dielectric Slab

Finally we present a numerical experiment from the published literature [3]. It concerns a dielectric slab, shown in Fig. 6, with $\varepsilon_{r1} = 1.44$ and $\varepsilon_{r2} = 2.56$. The exact dimensions can be found in [3], but to give an idea: $L \approx 5\lambda_0$. In [3] the RCS of this object is computed with a fast iterative method, using 206,200 unknowns. The total computation time was 38 hours. The result is shown in Fig. 7.

We analyzed the same object using MS-CBD. Fig. 8 shows our result. It corresponds well with that of [3]. The performance parameters of our analysis are summarized in Table II. It is noteworthy that the result has almost converged with SVD threshold $\tau=10^{-2}$. As Table II shows the computational effort grows significantly for decreasing τ, so it is worthwhile to use as large a value of τ as possible.

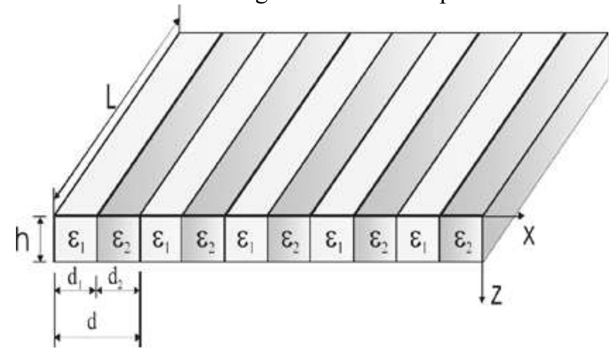

Fig. 6. Dielectric slab from reference [3].

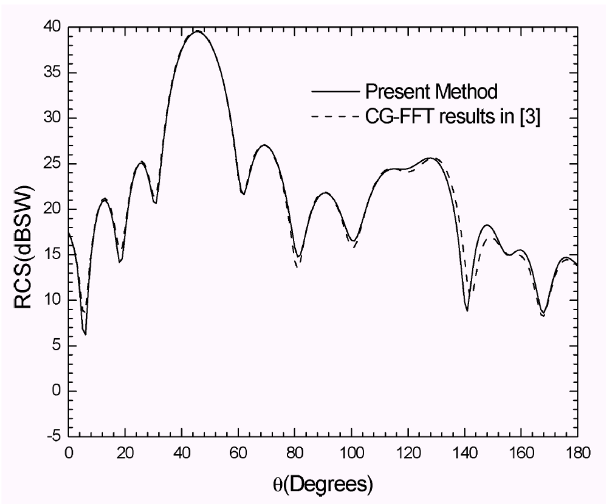

Fig. 7. RCS of dielectric slab calculated in reference [3].

Table II. Performance of MS-CBD on slab from reference [3]

N	210,711	
MS-CBD levels	9	
SVD threshold τ	10^{-2}	10^{-3}
Matrix build time	7,846 s	21,148 s
MS-CBD time	1,208 s	2,545 s
Matrix size	5.2 GB	7.2 GB
MS-CBD size	5.2 GB	7.2 GB
Backsub. time	22 s	25 s

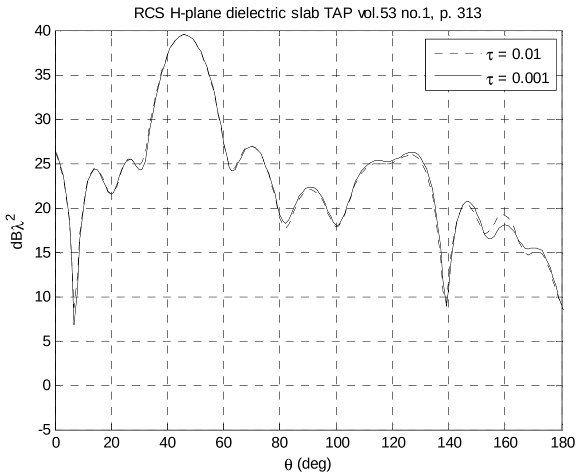

Fig. 8. RCS of dielectric slab from reference [3], computed with MS-CBD, using two different SVD thresholds.

VI. Conclusion

We have investigated the performance of the VIE in combination with the MS-CBD algorithm. Numerical experiments show a somewhat better complexity than the theoretical prediction of N^2. Furthermore we show that the efficiency does not depend on the permittivity of the object under study. Although not demonstrated in this paper, one of the most important features of the MS-CBD is its capacity to solve for arbitrarily many excitations without much extra cost. In the near future we will implement in our software the interaction with metal surfaces which will greatly enhance its scope.

References

[1] C. C. Lu, "A fast algorithm based on volume integral equation for analysis of arbitrarily shaped dielectric radomes," *IEEE Trans. Antennas Propag.*, vol. 51, no. 9, pp. 606–612, Sep. 2003.

[2] Z. Q. Zhang and Q. H. Liu, "A volume adaptive integral method (VAIM) for 3-D inhomogeneous objects," *IEEE AntennasWireless Propag. Lett.*, vol. 1, pp. 102–105, Jul. 2002

[3] X. Nie, L-W. Li, N. Yuan, T. S. Yeo and Y. Gan, "Precorrected-FFT Solution of the Volume Integral Equation for 3-D Inhomogeneous Dielectric Objects," *IEEE Trans. Antennas Propag*, vol. 53, no. 1, pp. 313-320, Jan. 2005

[4] A. Heldring, J. M. Rius, J. M. Tamayo, J. Parrón, E. Ubeda, 'Fast Direct Solution of Method of Moments Linear System,' *IEEE Trans. Antennas Propag*, Vol 55, No 11, pp. 3220-3228, Nov 2007

[5] A. Heldring, J. M. Rius, J. M. Tamayo, J. Parrón, E. Ubeda, 'Multiscale Compressed Block Decomposition for Fast Direct Solution of Method of Moments Linear System,' submitted to *IEEE Trans. on Antennas Propag.*

[6] M. Bebendorf, 'Approximation of boundary element matrices,' *Numer. Math*. (2000) 86: 565—589

[7] D.H. Schaubert, D.R. Wilton and A.W. Glisson, 'A Tetrahedral Modeling Method for Electromagnetic Scattering by Arbitrarily Shaped Inhomogeneous Dielectric Bodies,' *IEEE Trans. Antennas Propag*, Vol. AP-32, No. 1, pp 77-85, Jan 1984.